Language: **English**

Sunday, June 23, 2013

Problem 1. Find all ordered pairs (a, b) of positive integers for which the numbers $\dfrac{a^3b-1}{a+1}$ and $\dfrac{b^3a+1}{b-1}$ are both positive integers.

Problem 2. Let ABC be an acute triangle with $AB < AC$ and O be the center of its circumcircle ω. Let D be a point on the line segment BC such that $\angle BAD = \angle CAO$. Let E be the second point of intersection of ω and the line AD. If M, N and P are the midpoints of the line segments BE, OD and AC, respectively, show that the points M, N and P are collinear.

Problem 3. Show that

$$\left(a+2b+\frac{2}{a+1}\right)\left(b+2a+\frac{2}{b+1}\right) \geq 16$$

for all positive real numbers a and b such that $ab \geq 1$.

Problem 4. Let n be a positive integer. Two players, Alice and Bob, are playing the following game:

- Alice chooses n real numbers, not necessarily distinct
- Alice writes all pairwise sums on a sheet of paper and gives it to Bob (there are $\frac{n(n-1)}{2}$ such sums, not necessarily distinct)
- Bob wins if he finds correctly the initial n numbers chosen by Alice with only one guess

Can Bob be sure to win for the following cases?

a. $n = 5$ **b.** $n = 6$ **c.** $n = 8$

Justify your answer(s).

[For example, when $n = 4$, Alice may choose the numbers 1, 5, 7, 9, which have the same pairwise sums as the numbers 2, 4, 6, 10, and hence Bob cannot be sure to win.]

Time: 4 hours and 30 minutes
Each problem is worth 10 points